

SPIN REPRESENTATIONS OF REAL REFLECTION GROUPS OF NON-CRYSTALLOGRAPHIC ROOT SYSTEMS

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ABSTRACT. A uniform parametrization for the irreducible spin representations of Weyl groups in terms of nilpotent orbits is recently achieved by Ciubotaru (2011). This paper is a generalization of this result to other real reflection groups.

Let $(V_0, R, V_0^\vee, R^\vee)$ be a root system with the real reflection group W . We define a special subset of points in V_0^\vee which will be called solvable points. Those solvable points, in the case R crystallographic, correspond to the nilpotent orbits whose elements have a solvable centralizer in the corresponding Lie algebra. Then a connection between the irreducible spin representations of W and those solvable points in V_0^\vee is established.

1. INTRODUCTION

1.1. Introduction. Let $(V_0, R, V_0^\vee, R^\vee)$ be a reduced root system (see section 2.1) with the corresponding real reflection group W . Let $O(V_0^\vee)$ be the orthogonal group of V_0^\vee and let $\text{Pin}(V_0^\vee)$ be the double cover of $O(V_0^\vee)$ with the covering map p (see Section 2.2). Since W is an orthogonal transformation on V_0^\vee , we may consider the preimage $p^{-1}(W)$, the double cover \widetilde{W} of W . The classification of genuine irreducible representations of \widetilde{W} (i.e. representations which do not factor through W), or so-called spin representations of W , has been known for a long time from the work of Schur, Morris, Read, Stembridge and others ([7], [8], [9], [10] and [17]). A recent paper [4] by Ciubotaru introduces an approach to classify genuine representations in the framework of nilpotent orbits in semisimple Lie algebras. The orbits of nilpotent elements whose elements have a solvable centralizer play an essential role in the classification. This paper generalizes [4] to the reflection groups associated to noncrystallographic root systems.

An important ingredient in our paper is a special subset of points in V_0^\vee , which will be called solvable points. For maximum generality, fix a W -invariant parameter function $c : R \rightarrow \mathbb{R}$ and write c_α for $c(\alpha)$. Before giving a precise definition of solvable points, we define a point $\gamma \in V_0^\vee$ to be *distinguished* if

$$|\{\alpha \in R : (\alpha, \gamma) = c_\alpha\}| = |\{\alpha \in R : (\alpha, \gamma) = 0\}| + \dim_{\mathbb{R}} V_0.$$

This combinatorial definition is introduced in [12] by Heckman and Opdam and can be viewed as a generalization of the characterization of distinguished nilpotent orbits in the Bala-Carter theory ([3]). Let Δ be a fixed set of simple roots in R . For $J \subseteq \Delta$, let $V_{0,J}^\vee$ be the real vector space spanned by coroots corresponding to J . We extend the set of distinguished points to a larger class:

Definition 1.1. A point γ in V_0^\vee is said to be *solvable* if γ is conjugate to some point which is distinguished in $V_{0,J}^\vee$ for some $J \subseteq \Delta$, and satisfies

$$|\{\alpha \in R : (\alpha, \gamma) = c_\alpha\}| = |\{\alpha \in R : (\alpha, \gamma) = 0\}| + |J|.$$

Let \mathcal{V}_{sol} be the set of W -orbits of solvable points in V_0^\vee .

It is easy to see that a solvable point with $J = \Delta$ is distinguished. The terminology of solvable points is explained by the following result, which will be proven in Section 3:

Theorem 1.2. *Let R be a crystallographic root system and set $c \equiv 2$. Let \mathfrak{g} be the semisimple Lie algebra associated to R . Let \mathcal{N}_{sol} be the set of nilpotent orbits in \mathfrak{g} whose elements have a solvable centralizer. Then there is a natural one-to-one correspondence between the set \mathcal{V}_{sol} and the set \mathcal{N}_{sol} .*

Roughly, the correspondence in Theorem 1.2 takes a nilpotent orbit to the semisimple element of a corresponding Jacobson-Morozov triple.

Our main result Theorem 1.3 below establishes a connection between those solvable points and the spin representations of W in the case that R is noncrystallographic. Before stating the main result, we need few more notations. Fix a symmetric W -invariant bilinear form \langle, \rangle on V_0^\vee as in (2.2). Define a Casimir-type element in the Clifford algebra $C(V_0^\vee)$:

$$\Omega_{\widetilde{W}} = -\frac{1}{4} \sum_{\alpha > 0, \beta > 0, s_\alpha(\beta) < 0} c_\alpha c_\beta |\alpha^\vee| |\beta^\vee| f_\alpha f_\beta,$$

where $s_\alpha \in W$ is the reflection corresponding to α and f_α is a certain element in $p^{-1}(s_\alpha)$ (see Section 2.2) and $|\alpha^\vee| = \langle \alpha^\vee, \alpha^\vee \rangle^{1/2}$. The element $\Omega_{\widetilde{W}}$ is introduced in [4] and is related to the Dirac operator for the graded affine Hecke algebra in [1] by Barbasch, Ciubotaru and Trapa. Indeed, $\Omega_{\widetilde{W}}$ is in the center of $C(V_0^\vee)$ and so a version of the Schur's lemma implies that this element acts on a simple \widetilde{W} -representation $(\widetilde{U}, \widetilde{\chi})$ by a scalar $\widetilde{\chi}(\Omega_{\widetilde{W}})$. Let $\text{Irr}_{\text{gen}}(\widetilde{W})$ be the set of irreducible genuine \widetilde{W} -representations. Define an equivalence relation \sim on $\text{Irr}_{\text{gen}}(\widetilde{W})$: $\widetilde{\sigma} \sim \widetilde{\sigma} \otimes \text{sgn}$, where sgn is the sign W -representation.

In Section 4, we prove our main result:

Theorem 1.3. *Let $(V_0, R, V_0^\vee, R^\vee)$ be a noncrystallographic root system. Fix a symmetric W -invariant bilinear form \langle, \rangle on V_0^\vee as in (2.2). Then there exists a unique surjective map*

$$\Phi : \text{Irr}_{\text{gen}}(\widetilde{W}) / \sim \rightarrow \mathcal{V}_{\text{sol}}$$

such that for any representative $\gamma \in \Phi([\widetilde{\chi}])$,

$$(1.1) \quad \widetilde{\chi}(\Omega_{\widetilde{W}}) = \langle \gamma, \gamma \rangle.$$

Furthermore, Φ is bijective if and only if $c_\alpha \neq 0$ for some $\alpha \in R$, and either:

- (1) $R = I_2(n)$ (n odd) or H_3 ; or
- (2) $R = I_2(n)$ (n even) with $\cos(k\pi/n)c' - \cos(l\pi/n)c'' \neq 0$ for any integers k, l which have distinct parity and $\cos(k\pi/n), \cos(l\pi/n) \neq \pm 1$, where c' and c'' are the two values corresponding to the two distinct W -orbits in the parameter function c .

We see from the expression of $\Omega_{\widetilde{W}}$ that $(\text{sgn} \otimes \widetilde{\chi})(\Omega_{\widetilde{W}}) = \widetilde{\chi}(\Omega_{\widetilde{W}})$ and this explains why we need the equivalence \sim in defining the surjective map above. Although the map Φ in H_4 or some special cases for $I_2(\text{even})$ fails to be bijective, the sizes of the fibers of Φ are still one most of time.

For the crystallographic cases (with the equal parameter $c \equiv 2$), we could still obtain a surjective map as the one in Theorem 1.3 by using our Theorem 1.2 to replace the image of the surjective map in [4, Theorem 1] with the set of solvable points. This explains how our result generalizes [4]. However, a surjective map with only the property (1.1) is not unique in general in the crystallographic cases.

Our motivation for the Theorem 1.3 is to study the Dirac cohomology for the graded affine Hecke algebra in the noncrystallographic cases by analogue with the crystallographic cases in [1] and [5]. We expect that those solvable points afford the central characters of interesting tempered modules with nonzero Dirac cohomology (in the sense of [1]). Theorem 1.3 is evidence for the claim, but some further information such as the W -module structures of tempered modules is needed. In view of the paper [5] by Ciubotaru and Trapa, perhaps a preliminary step to understand the Dirac cohomology is to look at the elliptic representation theory of W ([9]). A study of the latter object is carried out in Section 5.

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2. PRELIMINARIES

2.1. Root systems and notations. Fix a reduced root system $\Sigma = (V_0, R, V_0^\vee, R^\vee)$ over \mathbb{R} such that

- (1) $R \subset V_0$ and $R^\vee \subset V_0^\vee$ span the real vector spaces V_0 and V_0^\vee respectively.
- (2) There exists a bilinear pairing

$$(\cdot, \cdot) : V_0 \times V_0^\vee \rightarrow \mathbb{R},$$

and a bijection from R to R^\vee , denoted $\alpha \mapsto \alpha^\vee$, such that $(\alpha, \alpha^\vee) = 2$ for any $\alpha \in R$.

- (4) For $\alpha \in R$, the reflections

$$\begin{aligned} s_\alpha : V_0 &\rightarrow V_0, & s_\alpha(v) &= v - (v, \alpha^\vee)\alpha, \\ s_\alpha^\vee : V_0^\vee &\rightarrow V_0^\vee, & s_\alpha^\vee(v') &= v' - (\alpha, v')\alpha^\vee \end{aligned}$$

leave R and R^\vee invariant respectively.

- (5) For $\alpha \in R$, the only multiples of α in R are α and $-\alpha$. Moreover, $0 \notin R$ and $0 \notin R^\vee$.

A root system Σ (or simply R) is said to be *crystallographic* if $(\alpha, \beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in R$. In the literature, a root system is often by definition crystallographic and our terminology of root systems here is not quite standard.

Our primary concern in this paper is noncrystallographic cases. This includes $I_2(n)$ for $n = 5$ and $n \geq 7$, H_3 and H_4 and their corresponding Dynkin diagrams are as follows respectively:

$$\begin{array}{ccccccc} \alpha_1 & & \alpha_2 & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 \\ \circ & \text{---} & \circ & & \circ & \text{---} & \circ & \text{---} & \circ & & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ & n & & & 5 & & 5 & & & & 5 & & & & & & \end{array}$$

When $I_2(n)$ is mentioned later, we do not necessarily assume $I_2(n)$ to be noncrystallographic i.e. n can be any integer greater than or equal to 3. Our results naturally cover all $I_2(n)$. When $R = I_2(n), H_3, H_4$, we shall fix a W -invariant inner product $\langle \cdot, \cdot \rangle$ on V_0^\vee such that

$$(2.2) \quad \langle \alpha^\vee, \alpha^\vee \rangle = 2$$

for all $\alpha^\vee \in R^\vee$. Let $V = \mathbb{C} \otimes_{\mathbb{R}} V_0$ and let $V^\vee = \mathbb{C} \otimes_{\mathbb{R}} V_0^\vee$. Then we extend $\langle \cdot, \cdot \rangle$ to a symmetric W -invariant \mathbb{C} -bilinear form on V^\vee .

Denote by $W(R)$ or simply W the subgroup of $GL(V_0)$ generated by all the reflections s_α , where $\alpha \in R$. The map $s_\alpha \mapsto s_\alpha^\vee$ gives an embedding of W into $GL(V_0^\vee)$ so that $(v, w \cdot \omega) = (w \cdot v, \omega)$ for all $w \in W$, $v \in V_0$ and $\omega \in V_0^\vee$.

Fix a set R_+ of positive roots in R . Write $\alpha > 0$ for $\alpha \in R_+$ and write $\alpha < 0$ for $\alpha \in R \setminus R_+$. Let $R_+^\vee = \{\alpha^\vee : \alpha \in R_+\}$. Let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots in R_+ , where $r = \dim_{\mathbb{R}} V_0$.

The Coxeter group $W(I_2(n))$ is the dihedral group of order $2n$. The generators $s_{\alpha_1}, s_{\alpha_2}$ are subject to the relation: $(s_{\alpha_1} s_{\alpha_2})^n = 1$. The Coxeter group $W(H_3)$ has order 120 and is generated by 3 simple reflections, namely $s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}$, subject to the following relations:

$$(s_{\alpha_1} s_{\alpha_2})^5 = (s_{\alpha_2} s_{\alpha_3})^3 = (s_{\alpha_1} s_{\alpha_3})^2 = 1.$$

The Coxeter group $W(H_4)$ has order 14400 and is generated by 4 simple simple reflections, $s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_4}$ with the following relations:

$$(s_{\alpha_1} s_{\alpha_2})^5 = (s_{\alpha_2} s_{\alpha_3})^3 = (s_{\alpha_3} s_{\alpha_4})^3 = 1$$

and

$$(s_{\alpha_i} s_{\alpha_j})^2 = 1 \text{ for } |i - j| > 1.$$

2.2. The Clifford algebra and the Pin group. Fix an inner product $\langle \cdot, \cdot \rangle$ on V_0^\vee as in (2.2). Let $T(V_0^\vee)$ be the tensor algebra of V_0^\vee . Let I be the ideal generated by all the elements of the form

$$\omega \otimes \omega' + \omega' \otimes \omega + 2\langle \omega, \omega' \rangle.$$

Then the real Clifford algebra, denoted $C(V_0^\vee)$, with respect to V_0^\vee and the inner product $\langle \cdot, \cdot \rangle$ is the quotient algebra $T(V_0^\vee)/I$.

Let t be the anti-automorphism of $C(V_0^\vee)$ characterized by the properties:

$$(ab)^t = b^t a^t \text{ for } a, b \in C(V_0^\vee), \omega^t = -\omega, \text{ for } \omega \in V_0^\vee.$$

Define a map $\epsilon : V_0^\vee \rightarrow C(V_0^\vee)$, $\epsilon(\omega) = -\omega$. This induces an automorphism, still denoted by ϵ , from $C(V_0^\vee)$ to $C(V_0^\vee)$. The map ϵ decomposes $C(V_0^\vee)$ into $+1$ eigenspace $C(V_0^\vee)_{\text{even}}$ and -1 eigenspace $C(V_0^\vee)_{\text{odd}}$:

$$C(V_0^\vee) = C(V_0^\vee)_{\text{even}} \oplus C(V_0^\vee)_{\text{odd}}.$$

This induces a \mathbb{Z}_2 -grading on $C(V_0^\vee)$.

Define the Pin group of V_0^\vee to be

$$\text{Pin}(V_0^\vee) := \{a \in C(V_0^\vee) : \epsilon(a)V_0^\vee a^{-1} \subset V_0^\vee, a^t = a^{-1}\}.$$

We get a surjective homomorphism p from $\text{Pin}(V_0^\vee)$ to $O(V_0^\vee)$ defined by

$$p(a)\omega = \epsilon(a)\omega a^{-1}.$$

Here $O(V_0^\vee)$ is the orthogonal group of V_0^\vee with respect to the inner product \langle, \rangle .

Then we have the following exact sequence:

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Pin}(V_0^\vee) \xrightarrow{p} O(V_0^\vee) \rightarrow 1.$$

Let $\det_{V_0^\vee} : O(V_0^\vee) \rightarrow \{\pm 1\}$ be the determinant function on $O(V_0^\vee)$. Regarding W as a subgroup of $O(V_0^\vee)$, let $W_{\text{even}} = \ker(\det_{V_0^\vee}) \cap W$. For the notational convenience later, set

$$W' = \begin{cases} W & \text{if } \dim_{\mathbb{R}} V_0 \text{ is odd} \\ W_{\text{even}} & \text{if } \dim_{\mathbb{R}} V_0 \text{ is even} \end{cases}.$$

Define

$$\widetilde{W} = p^{-1}(W) \quad \text{and} \quad \widetilde{W}' = p^{-1}(W').$$

We describe the structure of \widetilde{W} . For $\alpha \in R$, define f_α to be the element in $C(V_0^\vee)$ such that

$$f_\alpha = \frac{\alpha^\vee}{|\alpha^\vee|}.$$

Then one sees that $p(f_\alpha) = s_\alpha$ and $f_\alpha^2 = -1$. Furthermore,

$$f_\alpha f_\beta f_\alpha = f_{s_\alpha(\beta)}.$$

The group \widetilde{W} is generated by all f_α for $\alpha \in \Delta$.

2.3. The double covers \widetilde{W} . In this subsection, we review representations of \widetilde{W} for non-crystallographic root systems.

Case of $W = W(I_2(n))$

The double cover \widetilde{W} has generators $f_{\alpha_1}, f_{\alpha_2}$ (notation in Section 2.2) and is determined by the relations:

$$(f_{\alpha_1} f_{\alpha_2})^n = f_{\alpha_1}^2 = f_{\alpha_2}^2 = -1.$$

The character table (including genuine and nongenuine representations) of $\widetilde{I_2(n)}$ is provided in Table 1 and 2 for the completeness.

Case of $W = W(H_3)$

It is well-known the group $W(H_3)$ can be realized as $\text{Alt}_5 \times \mathbb{Z}_2$ such that the longest element w_0 is identified with the order 2 element in \mathbb{Z}_2 . Here Alt_5 is the alternating group on 5 letters. Since there are 5 irreducible representations for Alt_5 , W has 10 irreducible representations.

We describe the group structure of the double cover $\widetilde{W(H_3)}$. Let \widetilde{H} be the subgroup generated by $f_{\alpha_1} f_{\alpha_2}$ and $f_{\alpha_2} f_{\alpha_3}$. Then \widetilde{H} is isomorphic to the double cover $\widetilde{\text{Alt}_5} = p^{-1}(\text{Alt}_5)$

TABLE 1. Character table of $\widetilde{W}(I_2(n))$ (n odd)

Characters	1	-1	f_{α_1}	$-f_{\alpha_1}$	$(f_{\alpha_1}f_{\alpha_2})^k$ ($k = 1, \dots, \frac{n-1}{2}$)	$-(f_{\alpha_1}f_{\alpha_2})^k$ ($k = 1, \dots, \frac{n-1}{2}$)
triv	1	1	1	1	1	1
sgn	1	1	-1	-1	1	1
ϕ_i ($i = 1, \dots, \frac{n-1}{2}$)	2	2	0	0	$2 \cos \frac{2ik\pi}{n}$	$2 \cos \frac{2ik\pi}{n}$
$\tilde{\chi}_1$	1	-1	$-\sqrt{-1}$	$\sqrt{-1}$	$(-1)^k$	$-(-1)^k$
$\tilde{\chi}_2$	1	-1	$\sqrt{-1}$	$-\sqrt{-1}$	$(-1)^k$	$-(-1)^k$
$\tilde{\rho}_i$ ($i = 1, \dots, \frac{n-1}{2}$)	2	-2	0	0	$2(-1)^k \cos \frac{2ik\pi}{n}$	$2(-1)^{k+1} \cos \frac{2ik\pi}{n}$

TABLE 2. Character table of $\widetilde{W}(I_2(n))$ (n even)

Characters	1	-1	f_{α_1}	f_{α_2}	$(f_{\alpha_1}f_{\alpha_2})^k$ ($k = 1, \dots, \frac{n-2}{2}$)	$-(f_{\alpha_1}f_{\alpha_2})^k$ ($k = 1, \dots, \frac{n-2}{2}$)	$(f_{\alpha_1}f_{\alpha_2})^{n/2}$
triv	1	1	1	1	1	1	1
sgn	1	1	-1	-1	1	1	1
σ_{α_1}	1	1	-1	1	$(-1)^k$	$(-1)^k$	$(-1)^{n/2}$
σ_{α_2}	1	1	1	-1	$(-1)^k$	$(-1)^k$	$(-1)^{n/2}$
ϕ_i ($i = 1, \dots, \frac{n-2}{2}$)	2	2	0	0	$2 \cos \frac{2ik\pi}{n}$	$2 \cos \frac{2ik\pi}{n}$	$2(-1)^i$
$\tilde{\rho}_i$ ($i = 1, \dots, \frac{n}{2}$)	2	-2	0	0	$2 \cos \frac{(2i-1)k\pi}{n}$	$-2 \cos \frac{(2i-1)k\pi}{n}$	0

of Alt_5 . Moreover, for a lift $\tilde{w}_0 \in \widetilde{W}$ of w_0 , \tilde{w}_0 is in the center of \widetilde{W} and $\tilde{w}_0^2 = -1$. The double cover \widetilde{W} is generated by \tilde{H} and \tilde{w}_0 .

Under this identification, the representations of W and \widetilde{W} can be deduced from that of Alt_5 and $\widetilde{\text{Alt}}_5$ respectively. The latter two objects are well known (see, for example, in [9, Table 1]). We provide the representations of Alt and $\widetilde{\text{Alt}}_5$ in Table 3 and Table 4 for completeness. For each representation ϕ of Alt_5 , there are two corresponding representations of W , denoted by ϕ^+ and ϕ^- such that ϕ^+ and ϕ^- are determined by

$$\phi^+(h) = \phi^-(h) = \phi(h) \text{ for any } h \in H, \quad \phi^\pm(w_0) = \pm 1,$$

where $H = p(\tilde{H})$ is identified with Alt_5 .

Similarly, for each genuine representation $\tilde{\chi}$ of $\widetilde{\text{Alt}}_5$, denote by $\tilde{\chi}^+$ and $\tilde{\chi}^-$ the two representations of \widetilde{W} corresponding to $\tilde{\chi}$ such that

$$\tilde{\chi}^+(h) = \tilde{\chi}^-(h) = \tilde{\chi}(h) \text{ for any } h \in \tilde{H}, \quad \tilde{\chi}^\pm(\tilde{w}_0) = \pm \sqrt{-1}.$$

In particular, there are 8 genuine irreducible \widetilde{W} -representations.

Let $\tau = (1 + \sqrt{5})/2$ and $\bar{\tau} = (1 - \sqrt{5})/2$, which will be used in the following Table 3 and Table 4.

Case of $W = W(H_4)$

Recall the group structure of $\widetilde{W}(H_4)$ described in [9]. Let $G = \widetilde{\text{Alt}}_5$, and let $G' = G \times G$. Let $\theta : G' \rightarrow G'$ be an automorphism of order 2 such that $\theta(g_1, g_2) = (g_2, g_1)$. Then the double cover $\widetilde{W}(H_4)$ is isomorphic to the semidirect product $G' \rtimes \langle \theta \rangle$ and has order 28800. Let z be the unique nontrivial element in the center of $\widetilde{\text{Alt}}_5$. The center Z of

TABLE 3. Character table of Alt_5 (identified with H)

Characters	1	$s_{\alpha_1}s_{\alpha_2}$	$(s_{\alpha_1}s_{\alpha_2})^2$	$s_{\alpha_2}s_{\alpha_3}$	$s_{\alpha_1}s_{\alpha_3}$
ϕ_1	1	1	1	1	1
ϕ_4	4	-1	-1	1	0
ϕ_3	3	τ	$\bar{\tau}$	0	-1
$\bar{\phi}_3$	3	$\bar{\tau}$	τ	0	-1
ϕ_5	5	0	0	-1	1

TABLE 4. Character table of genuine representations of $\widetilde{\text{Alt}}_5$ (identified with \widetilde{H})

Characters	± 1	$\pm f_{\alpha_1}f_{\alpha_2}$	$\pm (f_{\alpha_1}f_{\alpha_2})^2$	$\pm f_{\alpha_2}f_{\alpha_3}$	$f_{\alpha_1}f_{\alpha_3}$
$\tilde{\chi}_2$	± 2	$\pm \bar{\tau}$	$\mp \tau$	± 1	0
$\tilde{\bar{\chi}}_2$	± 2	$\pm \tau$	$\mp \bar{\tau}$	± 1	0
$\tilde{\chi}_6$	± 6	∓ 1	± 1	0	0
$\tilde{\chi}_4$	± 4	± 1	∓ 1	∓ 1	0

$\widetilde{W(H_4)} = G' \rtimes \langle \theta \rangle$ is $\{1, (z, z, 1)\}$ and $W(H_4)$ is isomorphic to the quotient G'/Z . Since the character tables for $W(H_4)$ and its double cover is rather lengthy, we refer readers to [9, Table II(i), (ii), (iii)]. We shall denote $\tilde{\chi}_i$ ($i = 35, \dots, 54$) for the characters of the 20 genuine irreducible representations of \widetilde{W} and the order of $\tilde{\chi}_i$ follows the one in [9, Table II(ii)].

2.4. The spin modules. This subsection describes the spin modules for $C(V_0^\vee)$, which will be used to study the elliptic representations of W in Section 5. The detail can be found in, for example, [13, Section 2.2]. To construct a spin module, it is easier to start with the complex Clifford algebra $C(V^\vee)$ of V^\vee , which is similarly defined as the quotient of the tensor algebra $T(V^\vee)$ by the ideal generated by elements of the form $\omega \otimes \omega' + \omega' \otimes \omega + 2\langle \omega, \omega' \rangle$, for $\omega, \omega' \in V^\vee$. The real Clifford algebra $C(V_0^\vee)$ naturally embeds into $C(V^\vee)$ as a real subalgebra.

Assume first that $\dim_{\mathbb{C}} V^\vee = 2m$ is even. Let U and U^* be m -dimensional isotropic subspaces of V^\vee with respect to \langle, \rangle so that $V^\vee = U \oplus U^*$. The simple $C(V^\vee)$ -module S which has 2^m dimension is isomorphic to the exterior algebra $\wedge^\bullet U$ of U with the action of $C(V^\vee)$ on $\wedge^\bullet U$ determined by the follows:

$$u \cdot (u_1 \wedge \dots \wedge u_k) = u \wedge u_1 \wedge \dots \wedge u_k, \quad \text{for } u \in U.$$

$$u^* \cdot (u_1 \wedge \dots \wedge u_k) = \sum_{i=1}^k (-1)^i 2\langle u^*, u_i \rangle u_1 \wedge \dots \wedge \hat{u}_i \wedge \dots \wedge u_k, \quad \text{for } u^* \in U^*.$$

The restriction of S to $C(V_0^\vee)$ is still an irreducible complex module. The further restriction of S to the even part $C(V_0^\vee)_{\text{even}}$, however, splits into two inequivalent modules S^+ and S^- , each of which has dimension 2^{m-1} (the choice is arbitrary for the later convenience). Moreover, the restriction of S to the group \widetilde{W} is again irreducible and denoted by

(γ, S) . Similarly, the restriction of S^+ and S^- to the group \widetilde{W}' gives rise two inequivalent irreducible \widetilde{W}' -representations, denoted again S^+ and S^- respectively.

Assume $\dim_{\mathbb{C}} V^{\vee} = 2m + 1$ is odd. Let u_1, \dots, u_{2m+1} be an orthonormal basis for V^{\vee} . Let \overline{V} be the complex subspace spanned by u_1, \dots, u_{2m} and let \overline{U} be the complex subspace spanned by u_{2m+1} . Then $V^{\vee} = \overline{V} \oplus \overline{U}$. Let S be the spin module for $C(\overline{V})$ as in the case of even dimension. We may define that u_{2m+1} acts on S by i or $-i$ so that S turns into two inequivalent $C(V^{\vee})$ modules of dimension 2^m , denoted by S^+ and S^- (the choice is again arbitrary). These modules remain irreducible when restricted to $C(V_0^{\vee})$. Moreover, the further restriction of S^+ and S^- to \widetilde{W}' are still irreducible and are again denoted by S^+ and S^- .

The following lemma will be used in Section 5.

Lemma 2.1. *Let*

$$\wedge^{\pm} V = \sum_i (-1)^i \wedge^i V$$

as a virtual representation of \widetilde{W}' . Then in the representation ring of \widetilde{W}' ,

$$(S^+ - S^-) \otimes (S^+ - S^-)^* = \frac{2}{[W : W']} \wedge^{\pm} V,$$

where $(S^+ - S^-)^$ is the dual of $S^+ - S^-$ and $[W : W']$ is the index of W' in W .*

Proof. See the discussion in [2, Chapter II 6].

Q.E.D.

Remark 2.2. We shall follow Section 2.3 for the notation of representations. The spin module S in each noncrystallographic case is as follows:

$$\begin{aligned} I_2(n) \ (n \text{ odd}) : \widetilde{\rho}_{(n-1)/2}, \quad I_2(n) \ (n \text{ even}) : \widetilde{\rho}_{n/2}, \\ H_3 : \widetilde{\chi}_2^+ \text{ or } \widetilde{\chi}_2^-, \quad H_4 : \widetilde{\chi}_{36}. \end{aligned}$$

3. DISTINGUISHED POINTS AND SOLVABLE POINTS

3.1. Definitions of distinguished and solvable points. Recall some terminology from the introduction. Fix a W -invariant parameter function $c : R \rightarrow \mathbb{R}$ and write c_{α} for $c(\alpha)$.

Definition 3.1. [12, Definition 1.4] A point γ in V^{\vee} is said to be *distinguished* if it satisfies

$$|\{\alpha \in R : (\alpha, \gamma) = c_{\alpha}\}| = |\{\alpha \in R : (\alpha, \gamma) = 0\}| + |\Delta|.$$

(In fact, any distinguished points are also in V_0^{\vee} and so we may replace V^{\vee} by V_0^{\vee} in the above definition as in the introduction.)

For $J \subseteq \Delta$, let V_J (resp. V_J^{\vee}) be the complex subspace of V (resp. V^{\vee}) spanned by the simple roots (resp. coroots) in J and let $R_J = V_J \cap R$ (resp. $R_J^{\vee} = V_J^{\vee} \cap R^{\vee}$).

Definition 3.2. A point γ in V^{\vee} is said to be *nilpotent* if it is conjugate to a point $\gamma' \in V_J^{\vee}$ for some $J \subseteq \Delta$ such that γ' is a distinguished point in V_J^{\vee} or equivalently,

$$|\{\alpha \in R_J : (\alpha, \gamma') = c_{\alpha}\}| = |\{\alpha \in R_J : (\alpha, \gamma') = 0\}| + |J|.$$

We shall say that γ is a nilpotent point associated to J if we want to emphasize the role of J .

A point γ in V^\vee is said to be *solvable* if γ is nilpotent associated to J for some $J \subset \Delta$ and γ further satisfies

$$|\{\alpha \in R : (\alpha, \gamma) = c_\alpha\}| = |\{\alpha \in R : (\alpha, \gamma) = 0\}| + |J|.$$

Let \mathcal{V}_{sol} be the set of W -orbits of solvable points in V^\vee .

The terminology of solvable points is motivated by Theorem 3.4 below. We first introduce few notations. Assume R is crystallographic. Set $c_\alpha = 2$ for all $\alpha \in R$. Let \mathfrak{g} be the semisimple Lie algebra corresponding to R . Let G be the connected Lie group of adjoint type over \mathbb{C} with the Lie algebra \mathfrak{g} . Fix a Cartan subalgebra $\mathfrak{h} \cong V^\vee$. For a nilpotent orbit \mathcal{O} in \mathfrak{g} , let $e \in \mathcal{O}$ and let $\{e, h, f\}$ be a Jacobson-Morozov standard triple such that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

The semisimple element h depends on the choices involved. However, due to a result of Kostant, the adjoint orbit of h in \mathfrak{g} is independent of the choices. Thus we may associate to each nilpotent orbit \mathcal{O} a semisimple orbit, denoted by \mathcal{O}_h , and indeed this association is an injection ([6, Theorem 3.5.4]). Moreover, this association gives rise to the following important map:

$$\Pi : \text{set of nilpotent orbits in } \mathfrak{g} \rightarrow \text{set of } W\text{-orbits of nilpotent points in } V^\vee,$$

$$\mathcal{O} \mapsto \mathfrak{h} \cap \mathcal{O}_h.$$

Note that the intersection $\mathfrak{h} \cap \mathcal{O}_h$ above is indeed a single W -orbit. This follows from the standard fact that two elements in \mathfrak{h} is in the same W -orbit if and only if they are in the same adjoint orbit in \mathfrak{g} ([6, Theorem 2.2.4]).

From the classification of nilpotent orbits in the Bala Carter theory (see [6, Lemma 8.2.1, Theorem 8.2.12]), we have:

Lemma 3.3. *The map Π is a well-defined bijection.*

The following result says which nilpotent orbits are mapped to the W -orbits of solvable points via Π :

Theorem 3.4. *Let R be a crystallographic root system and set $c \equiv 2$. Let \mathcal{N}_{sol} be the set of nilpotent orbits in \mathfrak{g} whose elements have a solvable centralizer. Then $\Pi(\mathcal{N}_{\text{sol}}) = \mathcal{V}_{\text{sol}}$. In particular, there is a one-to-one correspondence between the set \mathcal{V}_{sol} and the set \mathcal{N}_{sol} .*

A proof of Theorem 3.4 is given after Proposition 3.5.

Before stating Proposition 3.5, we review few facts about Jacobson-Morosov triples (see [6, Chapter 3] for the details). Let $\{e, h, f\}$ be a Jacobson-Morosov triple. By the representation theory of \mathfrak{sl}_2 , we have a \mathbb{Z} -grading on \mathfrak{g} :

$$(3.3) \quad \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i,$$

where $\mathfrak{g}_i = \{Z \in \mathfrak{g} : [h, Z] = iZ\}$. Let $\mathfrak{l} = \mathfrak{g}_0$ and let $\mathfrak{n} = \bigoplus_{i>0} \mathfrak{g}_i$. Then the parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ of \mathfrak{g} is called the Jacobson-Morozov parabolic subalgebra corresponding to $\{e, h, f\}$. Let $\mathfrak{n}_{\text{even}} = \bigoplus_{i>0} \mathfrak{g}_{2i}$.

For a Lie subalgebra $\mathfrak{c} \subset \mathfrak{g}$, write $\mathfrak{g}^{\mathfrak{c}}$ for the centralizer of \mathfrak{c} in \mathfrak{g} i.e.

$$\mathfrak{g}^{\mathfrak{c}} = \{X \in \mathfrak{g} : [X, Z] = 0 \text{ for all } Z \in \mathfrak{c}\}.$$

For an element $e \in \mathfrak{g}$, write \mathfrak{g}^e for the centralizer of e in \mathfrak{g} .

By the representation theory of \mathfrak{sl}_2 , \mathfrak{g}^e is the sum of the highest weight spaces of \mathfrak{sl}_2 -modules. Hence,

$$\mathfrak{g}^e = \bigoplus_{i \geq 0} \mathfrak{g}_i^e.$$

Using the representation theory of \mathfrak{sl}_2 again, we could see that

$$\mathfrak{l}^e = \mathfrak{g}_0^e = \{X \in \mathfrak{g} : [X, Z] = 0 \text{ for any } Z \in \langle e, h, f \rangle\},$$

where $\langle e, h, f \rangle$ is the Lie subalgebra of \mathfrak{g} generated by e, h, f .

For a subset $J \subset \Delta$, let $\mathfrak{l}_J = \mathfrak{h} \oplus \sum_{\alpha \in R_J} \mathfrak{g}_\alpha$ be the Levi subalgebra associated to J , where \mathfrak{g}_α is the root space corresponding to α .

Proposition 3.5. *Let \mathcal{O} be a nilpotent orbit in \mathfrak{g} . Let e be an element in \mathcal{O} such that the minimal Levi subalgebra containing e is equal to \mathfrak{l}_J for some $J \subseteq \Delta$ and a Jacobson-Morozov triple $\{e, h, f\}$ is in \mathfrak{l}_J with $h \in \mathfrak{h}$. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ be the Jacobson-Morozov parabolic subalgebra corresponding to $\{e, h, f\}$. Then the following conditions are equivalent:*

- (1) $\mathcal{O} \in \mathcal{N}_{\text{sol}}$ (notation in Theorem 3.4).
- (2) $\mathfrak{l}^e = \mathfrak{g}^{\mathfrak{l}_J}$ (and so $\mathfrak{l}^e \subset \mathfrak{h}$).
- (3) $\dim_{\mathbb{C}} \mathfrak{l} / \mathfrak{g}^{\mathfrak{l}_J} = \dim_{\mathbb{C}} \mathfrak{n}_{\text{even}} / [\mathfrak{n}_{\text{even}}, \mathfrak{n}_{\text{even}}]$.
- (4) $|\{\alpha \in R : (\alpha, h) = 2\}| = |\{\alpha \in R : (\alpha, h) = 0\}| + |J|$.

Proof. (1) \Rightarrow (2): By [6, Lemma 3.7.3], \mathfrak{l}^e is reductive. Thus by our assumption (1), \mathfrak{l}^e is contained in some toral subalgebra. For notational simplicity, set $\mathfrak{c} = \mathfrak{l}^e$. Then $\mathfrak{g}^{\mathfrak{c}}$ is a Levi subalgebra containing e and so contains some conjugate of \mathfrak{l}_J . Then \mathfrak{c} is contained in some conjugate of $\mathfrak{g}^{\mathfrak{l}_J}$. Hence $\dim_{\mathbb{C}} \mathfrak{c} \leq \dim_{\mathbb{C}} \mathfrak{g}^{\mathfrak{l}_J}$. However, we also have $\mathfrak{g}^{\mathfrak{l}_J} = \mathfrak{g}_0^{\mathfrak{l}_J} \subseteq \mathfrak{l}^e = \mathfrak{c}$, where the first equality follows from $h \in \mathfrak{l}_J$, and the second inclusion follows from $e \in \mathfrak{l}_J$. This forces $\mathfrak{c} = \mathfrak{g}^{\mathfrak{l}_J}$. Note that $\mathfrak{g}^{\mathfrak{l}_J} \subset \mathfrak{h}$ and so is \mathfrak{l}^e .

(2) \Rightarrow (1): By [6, Lemma 3.7.3] again, $\mathfrak{g}^e = \mathfrak{l}^e \oplus \mathfrak{n}^e$. Then $\mathfrak{g}^e \subset \mathfrak{h} \oplus \mathfrak{n}$ and so \mathfrak{g}^e is solvable.

(2) \Rightarrow (3): Recall that $\mathfrak{g}_0 = \mathfrak{l}$. Then, by the representation theory of \mathfrak{sl}_2 , $\text{ad}_e(\mathfrak{l}) = \mathfrak{g}_2$. Thus we have $\dim_{\mathbb{C}} \mathfrak{l} - \dim_{\mathbb{C}} \mathfrak{l}^e = \dim_{\mathbb{C}} \mathfrak{g}_2$. Since $e \in \mathfrak{n}_{\text{even}}$, $\mathfrak{g}_4 = [\mathfrak{n}_{\text{even}}, \mathfrak{n}_{\text{even}}]$. Then we also have $\dim_{\mathbb{C}} \mathfrak{g}_2 = \dim_{\mathbb{C}} \mathfrak{n}_{\text{even}} / [\mathfrak{n}_{\text{even}}, \mathfrak{n}_{\text{even}}]$. Combining equations, we get $\dim_{\mathbb{C}} \mathfrak{l} - \dim_{\mathbb{C}} \mathfrak{l}^e = \dim_{\mathbb{C}} \mathfrak{n}_{\text{even}} / [\mathfrak{n}_{\text{even}}, \mathfrak{n}_{\text{even}}]$. Now we apply (2) to obtain (3).

(3) \Rightarrow (2): We have seen that $\mathfrak{g}^{\mathfrak{l}_J} \subseteq \mathfrak{l}^e$ when proving (1) \Rightarrow (2). Then use the last equality in the argument of (2) \Rightarrow (3) and apply (3) together to obtain (2).

(3) \Leftrightarrow (4) follows from the following three relations: $|J| = \dim_{\mathbb{C}} \mathfrak{h} - \dim_{\mathbb{C}} \mathfrak{g}^{\mathfrak{l}_J}$,

$$|\{\alpha \in R : (\alpha, h) = 2\}| = \dim_{\mathbb{C}} \mathfrak{g}_2 = \dim_{\mathbb{C}} \mathfrak{n}_{\text{even}} / [\mathfrak{n}_{\text{even}}, \mathfrak{n}_{\text{even}}],$$

$$|\{\alpha \in R : (\alpha, h) = 0\}| = \dim_{\mathbb{C}} \mathfrak{l} / \mathfrak{h} = \dim_{\mathbb{C}} \mathfrak{l} - \dim_{\mathbb{C}} \mathfrak{h}.$$

Q.E.D.

Proof of Theorem 3.4. The statement follows from Lemma 3.3 and the equivalent conditions (1) and (4) in Proposition 3.5.

Q.E.D.

3.2. Classification of solvable points. For the crystallographic cases (with the parameter function $c \equiv 2$), classifying solvable points is equivalent to classifying nilpotent orbits whose centralizer has abelian reductive part (Condition (2) of Proposition 3.5). The reductive part of the centralizers has been computed for a long time in the literature (see for example [6] and [3]). This goes back to the work of Springer and Steinberg for classical groups, and Alekseev and others for exceptional groups. Then a complete list of nilpotent orbits in \mathcal{N}_{sol} can be computed accordingly (also see tables in [4]).

We classify all the solvable points for noncrystallographic root systems in the following Proposition 3.6, based on the classification of distinguished points by Heckman and Opdam [12, Section 4].

Proposition 3.6. *Let R be a noncrystallographic root system. The set of solvable points in V^\vee is described below.*

(a) *Suppose $c_\alpha \neq 0$ for some $\alpha \in R$.*

(1) *Recall α_1 and α_2 form a basis for R . For $i = 0, 1, \dots, n-1$, let $\beta_i \in R_+$ (resp. $\beta_i^\vee \in R_+^\vee$) be defined by*

$$\sin \frac{\pi}{n} \beta_i = \sin \frac{(i+1)\pi}{n} \alpha_1 + \sin \frac{i\pi}{n} \alpha_2,$$

$$\text{(resp. } \sin \frac{\pi}{n} \beta_i^\vee = \sin \frac{(i+1)\pi}{n} \alpha_1^\vee + \sin \frac{i\pi}{n} \alpha_2^\vee \text{)}.$$

For $k = 1, \dots, \lfloor n/2 \rfloor$, let $\{\beta_{k-1}^, \beta_{n-k}^*\}$ be the basis in V_0^\vee dual to $\{\beta_{k-1}^\vee, \beta_{n-k}^\vee\}$ (i.e. $\langle \beta_{k-1}^*, \beta_{k-1}^\vee \rangle = \langle \beta_{n-k}^*, \beta_{n-k}^\vee \rangle = 1$ and $\langle \beta_{k-1}^*, \beta_{n-k}^\vee \rangle = \langle \beta_{n-k}^*, \beta_{k-1}^\vee \rangle = 0$).*

For $k = 1, 2, \dots, \lfloor n/2 \rfloor$, define $\gamma_k = c_{\beta_{k-1}} \beta_{k-1}^ + c_{\beta_{n-k}} \beta_{n-k}^*$.*

(i) *n odd: Further define $\gamma_{(n+1)/2} = \frac{1}{2} \beta_{(n-1)/2}^\vee$. Then a point γ is solvable if and only if γ is conjugate to some γ_k ($k = 1, \dots, (n+1)/2$). Those γ_k can be written explicitly as*

$$\gamma_k = \frac{c}{4 \sin \frac{2k-1}{2n} \pi \sin \frac{\pi}{2n}} (\alpha_1^\vee + \alpha_2^\vee)$$

where $c = c_\alpha$ for any $\alpha \in R$. Moreover, the point γ_k is distinguished if and only if $k \neq (n+1)/2$.

(ii) *n even: Then a point γ is solvable if and only if γ is conjugate to some γ_k ($k = 1, \dots, n/2$). Moreover, the point γ_k is distinguished if and only if*

$$\left(c_{\beta_{k-1}} + c_{\beta_{n-k}} \cos \frac{(2k-1)\pi}{n} \right) \left(c_{\beta_{k-1}} \cos \frac{(2k-1)\pi}{n} + c_{\beta_{n-k}} \right) \neq 0.$$

(2) *H_3, H_4 case: A point γ is solvable if and only if γ is distinguished. (See Table 6 and Table 9 in Section 4 for the list of distinguished points.)*

(b) Suppose $c_\alpha = 0$ for all $\alpha \in R$. A point γ is solvable if and only if $\gamma = 0$.

Proof. When $c_\alpha \equiv 0$, it is straightforward from the definition. We consider the case that c is not identically zero. Since all distinguished points are computed in [12, Section 4] by Heckman and Opdam, we could accordingly find out all the nilpotent points. Then straightforward computation could determine which nilpotent points are solvable. We only give the details in $I_2(n)$ with n odd. Recall $\Delta = \{\alpha_1, \alpha_2\}$ is the set of simple roots. Up to conjugation under W , there are three possibility for a subset J of Δ : \emptyset , $\{\alpha_1\}$ and Δ . When $J = \Delta$, the nilpotent points associated to Δ are $\gamma_1, \dots, \gamma_{(n-1)/2}$. Those points are distinguished and so solvable. When $J = \{\alpha_1\}$, the only nilpotent point associated to J is $b\alpha_1$, where $b = c_{\alpha_1}/2$. Then it is easy to check $b\alpha_1$ is solvable in V^\vee by definitions and is conjugate to $\gamma_{(n+1)/2}$. Finally, for $J = \emptyset$, the only nilpotent point associated to \emptyset is 0 which is not solvable in V^\vee . This completes the list of solvable points in $I_2(n)$, n odd.

Q.E.D.

Remark 3.7. When $R = I_2(n)$ with n even, the solvable points γ_k defined in Proposition 3.6 may not be distinct in general even the parameter function c is not identically zero. This also explains why the surjective map Φ defined later in the proof of Theorem 4.2 is not a bijection in some situation.

4. A CORRESPONDENCE BETWEEN IRREDUCIBLE GENUINE \widetilde{W} -MODULES AND SOLVABLE POINTS

In this section, we prove our main result. Recall some notation from the introduction. Let \mathcal{V}_{sol} be the set of W -orbits of solvable points in V^\vee . Let $\text{Irr}_{\text{gen}}(\widetilde{W})$ be the set of genuine irreducible representations of \widetilde{W} . Define an equivalence relation on $\text{Irr}_{\text{gen}}(\widetilde{W})$: $\tilde{\sigma}_1 \sim \tilde{\sigma}_2$ if and only if $\tilde{\sigma}_1 = \text{sgn} \otimes \tilde{\sigma}_2$, where sgn is the sign W -representation. Fix a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on V_0^\vee as in (2.2). Define an element in the Clifford algebra $C(V_0^\vee)$:

$$\Omega_{\widetilde{W}} = -\frac{1}{4} \sum_{\alpha > 0, \beta > 0, s_\alpha(\beta) < 0} c_\alpha c_\beta |\alpha^\vee| |\beta^\vee| f_\alpha f_\beta,$$

where $|\alpha^\vee| = \langle \alpha^\vee, \alpha^\vee \rangle^{1/2}$.

Lemma 4.1. *The element $\Omega_{\widetilde{W}}$ is in the center of $C(V_0^\vee)$.*

Proof. Let $\widetilde{R} = \{(\alpha, \beta) \in R_+ \times R_+ : s_\alpha(\beta) < 0\}$. To show $\Omega_{\widetilde{W}}$ is in the center, it suffices to check that for any simple root γ and any pair $(\alpha, \beta) \in \widetilde{R}$, $-f_\gamma(f_\alpha f_\beta) f_\gamma = f_{\alpha'} f_{\beta'}$ for some $(\alpha', \beta') \in \widetilde{R}$. To this end, we divide into four cases. If $s_\alpha(\beta) = -\gamma$, then consider $-f_\gamma(f_\alpha f_\beta) f_\gamma = -f_\gamma f_{-s_\alpha(\beta)} f_\alpha f_\gamma = f_\alpha f_\gamma$. Then $(\alpha, \gamma) \in \widetilde{R}$ as desired. If $\gamma = \alpha$, then $-f_\alpha(f_\alpha f_\beta) f_\alpha = f_\alpha f_{-s_\alpha(\beta)}$. Again $(\alpha, -s_\alpha(\beta)) \in \widetilde{R}$ as desired. For the case $\alpha = \beta$, we similarly have $-f_\beta(f_\alpha f_\beta) f_\beta = f_\alpha f_{-s_\alpha(\beta)}$. The remaining case is that $\gamma \neq \alpha$ and $\gamma \neq \beta$ and $s_\alpha(\beta) \neq -\gamma$. We have $-f_\gamma(f_\alpha f_\beta) f_\gamma = (f_\gamma f_\alpha f_\gamma)(f_\gamma f_\beta f_\gamma) = f_{s_\gamma(\alpha)} f_{s_\gamma(\beta)}$. Note that $(s_\gamma(\alpha), s_\gamma(\beta)) \in \widetilde{R}$ as desired since $s_{s_\gamma(\alpha)}(s_\gamma(\beta)) = s_\gamma(s_\alpha(\beta))$.

Q.E.D.

Now Lemma 4.1 and a version of Schur's lemma imply that $\Omega_{\widetilde{W}}$ acts on an irreducible \widetilde{W} -representation $(\widetilde{U}, \widetilde{\chi})$ by a scalar $\widetilde{\chi}(\Omega_{\widetilde{W}})$.

Theorem 4.2. *Let $(V_0, R, V_0^\vee, R^\vee)$ be a noncrystallographic root system. Fix a symmetric bilinear form \langle, \rangle on V_0^\vee as in (2.2). Then there exists a unique surjective map*

$$\Phi : \text{Irr}_{\text{gen}}(\widetilde{W}) / \sim \rightarrow \mathcal{V}_{\text{sol}}$$

such that for any $\widetilde{\chi} \in \text{Irr}_{\text{gen}}(\widetilde{W})$ and for any representative $\gamma \in \Phi([\widetilde{\chi}])$,

$$(4.4) \quad \widetilde{\chi}(\Omega_{\widetilde{W}}) = \langle \gamma, \gamma \rangle.$$

Furthermore, Φ is bijective if and only if $c_\alpha \neq 0$ for some $\alpha \in R$, and either:

- (1) $R = I_2(n)$ (n odd) or H_3 ; or
- (2) $R = I_2(n)$ (n even) with $\cos(k\pi/n)c' - \cos(l\pi/n)c'' \neq 0$ for any integers k, l with distinct parity and $\cos(k\pi/n), \cos(l\pi/n) \neq \pm 1$, where c' and c'' are the two values corresponding to the two distinct W -orbits in the parameter function c .

The proof of Theorem 4.2 will be done by case-by-case analysis. For $\widetilde{\chi} \in \text{Irr}_{\text{gen}}(\widetilde{W})$, set $a(\widetilde{\chi}) = \widetilde{\chi}(\Omega_{\widetilde{W}})$. To begin with, we write down a slightly more explicit formula for $a(\widetilde{\chi})$:

$$(4.5) \quad a(\widetilde{\chi}) = \frac{1}{4} \sum_{\alpha > 0} c_\alpha^2 |\alpha^\vee|^2 - \frac{1}{4} \sum_{(\alpha, \beta) \in \widetilde{R}} c_\alpha c_\beta |\alpha^\vee| |\beta^\vee| \frac{\text{tr}_{\widetilde{\chi}}(f_\alpha f_\beta)}{\dim_{\mathbb{C}} \widetilde{\chi}},$$

where $\text{tr}_{\widetilde{\chi}}(f_\alpha f_\beta)$ is the value of the character of $\widetilde{\chi}$ on $f_\alpha f_\beta$ and

$$\widetilde{R} = \{(\alpha, \beta) \in R_+ \times R_+ : s_\alpha(\beta) < 0, \alpha \neq \beta\}.$$

Type $I_2(n)$, n odd

Since there is only one W -orbit on R , we set $c = c_\alpha$ for simplicity. Recall from Proposition 3.6 that the solvable points are, for $k = 1, \dots, (n+1)/2$,

$$\gamma_k = \frac{c}{4 \sin \frac{2k-1}{2n} \pi \sin \frac{\pi}{2n}} (\alpha_1^\vee + \alpha_2^\vee).$$

The squares of their lengths are

$$\langle \gamma_k, \gamma_k \rangle = \frac{c^2}{2 \sin^2 \left(\frac{2k-1}{2n} \pi \right)}$$

for $k = 1, \dots, (n+1)/2$.

We next compute the value $a(\widetilde{\chi})$ for $\widetilde{\chi} \in \text{Irr}_{\text{gen}}(\widetilde{W})$. The following two formulas, whose proofs are elementary (but possibly lengthy), may be useful in computing $a(\widetilde{\chi})$:

$$(4.6) \quad \sum_{k=1}^A k \cos(rk) = -\frac{\sin^2(r(A+1)/2)}{2 \sin^2(r/2)} + (A+1) \frac{\sin r(1/2 + A)}{2 \sin(r/2)},$$

$$(4.7) \quad \sum_{k=1}^A \cos(rk) = -\frac{1}{2} + \frac{\sin r(1/2 + A)}{2 \sin(r/2)},$$

where A is a positive integer and $r \in \mathbb{R} \setminus \{2\pi n : n \in \mathbb{Z}\}$.

By a simple computation, there are $2n - 4k$ pairs of roots $(\alpha, \beta) \in \tilde{R}$ forming an angle $k\pi/n$ for $k = 1, \dots, (n-1)/2$. Here a pair of roots α, β forming an angle θ means that $\cos \theta = \frac{\langle \alpha^\vee, \beta^\vee \rangle}{|\alpha^\vee| |\beta^\vee|}$.

Then, by (4.5) and Table 1,

$$\begin{aligned} a(\tilde{\rho}_i) &= \frac{1}{2}c^2n - \frac{1}{2}c^2 \sum_{k=1}^{(n-1)/2} (2n-4k)(-1)^{k+1} \frac{2 \cos(2ik\pi/n)}{2} \\ &= \frac{1}{2}c^2n + \frac{1}{2}c^2 \sum_{k=1}^{(n-1)/2} (2n-4k) \cos((n-2i)k\pi/n) \\ &= \frac{c^2}{2 \sin^2\left(\frac{n-2i}{2n}\pi\right)}. \end{aligned}$$

The last equality can be deduced from (4.6) and (4.7) with some further simplification. Similarly,

$$a(\tilde{\chi}_1) = a(\tilde{\chi}_2) = \frac{1}{2}c^2n - \frac{1}{2}c^2 \sum_{k=1}^{(n-1)/2} (2n-4k)(-1)^{k+1} = \frac{1}{2}c^2.$$

Note that $[\tilde{\chi}_1]$ contains $\tilde{\chi}_1$ and $\tilde{\chi}_2$ and each $[\tilde{\rho}_i]$ contains only the character $\tilde{\rho}_i$. Now, by above computations, the bijection Φ defined by

$$[\tilde{\rho}_i] \xrightarrow{\Phi} W\gamma_{(n+1)/2-i} \quad (i = 1, \dots, (n-1)/2); \quad [\tilde{\chi}_1] \xrightarrow{\Phi} W\gamma_{(n+1)/2},$$

satisfies the desired property. Here $W\gamma$ means the W -orbit of γ in V_0^\vee .

Type $I_2(n)$, n even

In the case of n even, α_1 and α_2 are in distinct W -orbits and so the unequal parameters case may happen. For notational convenience, set $c_1 = c_{\alpha_1}$ and $c_2 = c_{\alpha_2}$. For $i = 0, 1, \dots, n-1$, let $\beta_i^\vee \in R_+^\vee$ defined as in Proposition 3.6. Then according to Proposition 3.6, for $k = 1, \dots, \frac{n}{2}$, the solvable points γ_k are determined by

$$(4.8) \quad \langle \beta_{n-k}^\vee, \gamma_k \rangle = c_{\beta_{n-k}}, \quad \langle \beta_{k-1}^\vee, \gamma_k \rangle = c_{\beta_{k-1}}.$$

Note that as n is even, $n-k$ and $k-1$ have different parity, and so $\{c_{\beta_{n-k}}, c_{\beta_{k-1}}\} = \{c_1, c_2\}$.

Now elementary computation gives

$$\langle \gamma_k, \gamma_k \rangle = \frac{1}{2 \sin^2\left(\frac{n-2k+1}{n}\pi\right)} \left(c_1^2 - 2c_1c_2 \cos \frac{n-2k+1}{n}\pi + c_2^2 \right).$$

Next step is to compute $a(\tilde{\chi})$. For $\alpha \in R$, denote the W -orbit of α by $W\alpha$. Again we record the number of pairs of roots $(\alpha, \beta) \in \tilde{R}$ forming certain angles θ :

TABLE 5. The set \tilde{R} of $I_2(n)$ (n even)

θ	number of pairs (α, β) in \tilde{R}
$2k\pi/n$ $k = 1, \dots, \lfloor n/4 \rfloor$	(1) $n - 4k$ for $\alpha, \beta \in W\alpha_1$ (2) $n - 4k$ for $\alpha, \beta \in W\alpha_2$ (3) 0 for α, β in distinct W -orbits.
$(2k-1)\pi/n$ $k = 1, \dots, \lfloor n/4 \rfloor$	(1) $2(n - 4k + 2)$ for α, β in distinct W -orbits (2) 0 for α, β in the same W -orbit

Regard $a(\widetilde{\rho}_i)$ as a homogeneous polynomial of degree 2 with indeterminants c_1 and c_2 . Then, the coefficient of c_1^2 is equal to

$$\frac{1}{2} \binom{n}{2} - \frac{1}{2} \sum_{k=1}^{\lfloor n/4 \rfloor} (n-4k)(-1)^k \frac{2 \cos \frac{2i-1}{n}(2k)\pi}{2} = \frac{1}{2 \sin^2 \left(\frac{2i-1}{n} \pi \right)}.$$

The last equality can be again deduced from the formulas (4.6) and (4.7). The coefficient of c_2^2 has the same formula as that of c_1^2 . It remains to compute the coefficient of $c_1 c_2$:

$$\begin{aligned} & \frac{1}{2} \sum_{k=1, k \text{ odd}}^{\lfloor n/2 \rfloor} 2(n-2k)(-1)^k \frac{2 \cos \frac{2i-1}{n} k \pi}{2} \\ &= \sum_{k=1}^{\lfloor n/2 \rfloor} (n-2k)(-1)^k \frac{2 \cos \frac{2i-1}{n} k \pi}{2} - \sum_{l=1}^{\lfloor n/4 \rfloor} (n-2(2l))(-1)^l \frac{2 \cos \frac{2i-1}{n} (2l) \pi}{2} \\ &= \left(\frac{n}{2} - \frac{1}{2 \sin^2 \left(\frac{2i-1}{2n} \pi \right)} \right) - 2 \left(\frac{n}{4} - \frac{1}{2 \sin^2 \left(\frac{2i-1}{n} \pi \right)} \right) \\ &= -\frac{\cos \frac{2i-1}{n} \pi}{\sin^2 \left(\frac{2i-1}{n} \pi \right)}. \end{aligned}$$

Hence, we obtain a surjection $\Phi : \text{Irr}_{\text{gen}}(\widetilde{W}) \rightarrow \mathcal{V}_{\text{sol}}$ defined by

$$[\rho_i] \xrightarrow{\Phi} W\gamma_{n/2-i+1} \quad (i = 1, \dots, n/2).$$

For the bijectivity of Φ , further calculation shows that γ_k in (4.8) are in distinct W -orbits if and only if the conditions stated in (2) of the theorem hold. We skip the details of the computation.

Indeed, if we have $\cos(p\pi/n)c_1 - \cos(q\pi/n)c_2 = 0$ for some p, q with distinct parities and $\cos(p\pi/n), \cos(q\pi/n) \neq \pm 1$, then two distinguished points γ_k will be in the same W -orbit. This causes the bijectivity fails. Interestingly, if we keep $\cos(p\pi/n)c_1 - \cos(q\pi/n)c_2 = 0$ for some p, q with distinct parities, but we now have $\cos(p\pi/n) = \pm 1$ or $\cos(q\pi/n) = \pm 1$, then the bijectivity still holds. However, one of the solvable points γ_k will become non-distinguished (Proposition 3.6).

Type H_3

Set $c = c_\alpha$ for all α . Let $\{\omega_1, \omega_2, \omega_3\}$ forms a basis for V_0^\vee dual to $\{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$ so that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. By Proposition 3.6, all solvable points are distinguished points. The distinguished points (from [12, Table 4.13]) and their lengths are listed in Table 6. We provide some data in Table 7 and Table 8 so that (4.5) can be used to compute $a(\widetilde{\chi})$. The explicit form of the bijection Φ can be read from Table 8. Recall $\tau = (1 + \sqrt{5})/2$ and $\bar{\tau} = (1 - \sqrt{5})/2$.

Type H_4

Set $c = c_\alpha$ for all α . Let $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ be a basis for V_0^\vee dual to $\{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, \alpha_4^\vee\}$ so that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. The distinguished points are shown in Table 9. Relevant data in computing $a(\widetilde{\chi})$ are given in Table 10 and Table 11. The characters in Table 11 are listed

TABLE 6. Distinguished/Solvable points in H_3

Labels	distinguished/solvable points γ	$\langle \gamma, \gamma \rangle$
γ_1	$c\omega_1 + c\omega_2 + c\omega_3$	$(\frac{43}{2}\tau + \frac{19}{2}\bar{\tau})c^2$
γ_2	$(1 + \tau)^{-1}(c\omega_1 + c\omega_2 + c\tau\omega_3)$	$\frac{11}{2}c^2$
γ_3	$(1 + \tau)^{-1}(c\omega_1 + c\omega_2 + c(1 + \tau)\omega_3)$	$8c^2$
γ_4	$(2 + 3\tau)^{-1}(c(1 + \tau)\omega_1 + c\tau\omega_2 + c\omega_3)$	$(\frac{19}{2}\tau + \frac{43}{2}\bar{\tau})c^2$

TABLE 7. The set \tilde{R} of H_3

θ	number of pairs (α, β) in \tilde{R} with $\cos \theta = \frac{\langle \alpha, \beta \rangle}{ \alpha \beta }$
$\pi/5$	36
$2\pi/5$	12
$\pi/3$	20

TABLE 8. $a(\tilde{\chi})$ of H_3

Characters $\tilde{\chi}$	$\dim_{\mathbb{C}} \tilde{\chi}$	$\text{tr}(-f_{\alpha_1}f_{\alpha_2})$	$\text{tr}(-(f_{\alpha_1}f_{\alpha_2})^2)$	$\text{tr}(-f_{\alpha_2}f_{\alpha_3})$	$a(\tilde{\chi})$
$\tilde{\chi}_2^+, \tilde{\chi}_2^-$	2	$-\bar{\tau}$	τ	-1	$\langle \gamma_4, \gamma_4 \rangle$
$\tilde{\chi}_2^-, \tilde{\chi}_2^+$	2	$-\tau$	$\bar{\tau}$	-1	$\langle \gamma_1, \gamma_1 \rangle$
$\tilde{\chi}_6^+, \tilde{\chi}_6^-$	6	1	-1	0	$\langle \gamma_2, \gamma_2 \rangle$
$\tilde{\chi}_4^+, \tilde{\chi}_4^-$	4	-1	1	1	$\langle \gamma_3, \gamma_3 \rangle$

in the same order as the ones in [9, Table II(ii)]. The explicit surjective map of Φ can be seen in Table 11. Recall $\tau = (1 + \sqrt{5})/2$ and $\bar{\tau} = (1 - \sqrt{5})/2$.

TABLE 9. Distinguished/Solvable points in H_4

Labels	distinguished/solvable points γ	$\langle \gamma, \gamma \rangle$
γ_1	$c\omega_1 + c\omega_2 + c\omega_3 + c\omega_4$	$(238\tau + 94\bar{\tau})c^2$
γ_2	$(1 + \tau)^{-1}(c\omega_1 + c\omega_2 + c\tau\omega_3 + c\omega_4)$	$(48\tau + 24\bar{\tau})c^2$
γ_3	$(1 + \tau)^{-1}(c\omega_1 + c\omega_2 + c\tau\omega_3 + c(1 + \tau)\omega_4)$	$(64\tau + 28\bar{\tau})c^2$
γ_4	$(1 + \tau)^{-1}(c\omega_1 + c\omega_2 + c(1 + \tau)\omega_3 + c(1 + \tau)\omega_4)$	$(90\tau + 42\bar{\tau})c^2$
γ_5	$(2 + 3\tau)^{-1}(c(1 + \tau)\omega_1 + c\tau\omega_2 + c\omega_3 + c(1 + 2\tau)\omega_4)$	$30c^2$
γ_6	$(2 + 3\tau)^{-1}(c(1 + \tau)\omega_1 + c\tau\omega_2 + c\omega_3 + c(1 + 3\tau)\omega_4)$	$36c^2$
γ_7	$(2 + 3\tau)^{-1}(c(1 + \tau)\omega_1 + c\tau\omega_2 + c\omega_3 + c(2 + 3\tau)\omega_4)$	$40c^2$
γ_8	$(3 + 5\tau)^{-1}(c(1 + 2\tau)\omega_1 + c\omega_2 + c\tau\omega_3 + c\tau\omega_4)$	$(42\tau + 90\bar{\tau})c^2$
γ_9	$(2 + 4\tau)^{-1}(c\omega_1 + c\tau\omega_2 + c\tau\omega_3 + c\omega_4)$	$17/2c^2$
γ_{10}	$(2 + 3\tau)^{-1}(c\omega_1 + c\omega_2 + c\tau\omega_3 + c\omega_4)$	$(24\tau + 48\bar{\tau})c^2$
γ_{11}	$(3 + 5\tau)^{-1}(c\tau\omega_1 + c\tau\omega_2 + c\omega_3 + c\tau\omega_4)$	$(28\tau + 64\bar{\tau})c^2$
γ_{12}	$(5 + 8\tau)^{-1}(c\omega_1 + c(1 + 2\tau)\omega_2 + c\omega_3 + c\tau\omega_4)$	$(94\tau + 238\bar{\tau})c^2$
γ_{13}	$(1 + 2\tau)^{-1}(c\omega_2 + c\tau\omega_3 + c\tau\omega_4)$	$16c^2$
γ_{14}	$(2 + 3\tau)^{-1}(c\tau\omega_2 + c\tau\omega_3 + c\omega_4)$	$(18\tau + 34\bar{\tau})c^2$
γ_{15}	$(1 + \tau)^{-1}(c\omega_1 + c\omega_2 + c(1 + \tau)\omega_4)$	$(34\tau + 18\bar{\tau})c^2$
γ_{16}	$(1 + 2\tau)^{-1}(c\omega_2 + c\tau\omega_3)$	$10c^2$
γ_{17}	$(1 + \tau)^{-1}c\omega_2$	$6c^2$

TABLE 10. The set \tilde{R} of H_4

θ	number of pairs (α, β) in \tilde{R} with $\cos \theta = \frac{\langle \alpha, \beta \rangle}{ \alpha \beta }$
$\pi/5$	432
$2\pi/5$	144
$\pi/3$	400

TABLE 11. $a(\tilde{\chi})$ of H_4

Characters $\tilde{\chi}$	$\dim_{\mathbb{C}} \tilde{\chi}$	$\text{tr}(-f_{\alpha_1} f_{\alpha_2})$	$\text{tr}(-(f_{\alpha_1} f_{\alpha_2})^2)$	$\text{tr}(-f_{\alpha_2} f_{\alpha_3})$	$a(\tilde{\chi})$
$\tilde{\chi}_{35}$	4	$-2\bar{\tau}$	2τ	-2	$\langle \gamma_{12}, \gamma_{12} \rangle$
$\tilde{\chi}_{36}$	4	-2τ	$2\bar{\tau}$	-2	$\langle \gamma_1, \gamma_1 \rangle$
$\tilde{\chi}_{37}$	12	2	-2	0	$\langle \gamma_{17}, \gamma_{17} \rangle$
$\tilde{\chi}_{38}$	8	-2	2	2	$\langle \gamma_{13}, \gamma_{13} \rangle$
$\tilde{\chi}_{39}$	16	$2\bar{\tau}$	-2τ	-2	$\langle \gamma_3, \gamma_3 \rangle$
$\tilde{\chi}_{40}$	16	2τ	$-2\bar{\tau}$	-2	$\langle \gamma_{11}, \gamma_{11} \rangle$
$\tilde{\chi}_{41}$	48	-2	2	0	$\langle \gamma_6, \gamma_6 \rangle$
$\tilde{\chi}_{42}$	32	2	-2	2	$\langle \gamma_9, \gamma_9 \rangle$
$\tilde{\chi}_{43}$	12	2	-2	0	$\langle \gamma_{17}, \gamma_{17} \rangle$
$\tilde{\chi}_{44}$	12	2	-2	0	$\langle \gamma_{17}, \gamma_{17} \rangle$
$\tilde{\chi}_{45}$	12	$-2\tau^2$	$2\bar{\tau}^2$	0	$\langle \gamma_4, \gamma_4 \rangle$
$\tilde{\chi}_{46}$	12	$-2\bar{\tau}^2$	$2\tau^2$	0	$\langle \gamma_8, \gamma_8 \rangle$
$\tilde{\chi}_{47}$	36	$2\bar{\tau}$	-2τ	0	$\langle \gamma_{15}, \gamma_{15} \rangle$
$\tilde{\chi}_{48}$	36	2τ	$-2\bar{\tau}$	0	$\langle \gamma_{14}, \gamma_{14} \rangle$
$\tilde{\chi}_{49}$	24	-2τ	$2\bar{\tau}$	0	$\langle \gamma_2, \gamma_2 \rangle$
$\tilde{\chi}_{50}$	24	$-2\bar{\tau}$	2τ	0	$\langle \gamma_{10}, \gamma_{10} \rangle$
$\tilde{\chi}_{51}$	20	0	0	2	$\langle \gamma_{16}, \gamma_{16} \rangle$
$\tilde{\chi}_{52}$	20	0	0	2	$\langle \gamma_{16}, \gamma_{16} \rangle$
$\tilde{\chi}_{53}$	60	0	0	0	$\langle \gamma_5, \gamma_5 \rangle$
$\tilde{\chi}_{54}$	40	0	0	-2	$\langle \gamma_7, \gamma_7 \rangle$

Remark 4.3. In the crystallographic cases, Ciubotaru [4] constructed a surjective map Θ from $\text{Irr}_{\text{gen}}(\tilde{W})/\sim$ to \mathcal{N}_{sol} . Composing the map Θ and the bijection $\Pi : \mathcal{N}_{\text{sol}} \rightarrow \mathcal{V}_{\text{sol}}$ defined in Section 3.1, we would again obtain a surjective map as the one in Theorem 4.2.

5. ELLIPTIC REPRESENTATIONS OF W

5.1. Elliptic representation theory of finite groups. This section is based on [11, Section 3.1]. Let Γ be a finite group and let V be a Γ -representation over \mathbb{C} . Then an element $\gamma \in \Gamma$ is said to be *elliptic* if γ does not have any fixed point in V , or equivalently, $\det_V(\gamma - 1) \neq 0$, where \det_V is the determinant function for the linear operators on V . A conjugacy class C in Γ is called *elliptic* if C contains an elliptic element.

Let $R(\Gamma)$ be the representation ring of Γ over \mathbb{C} . Let \mathcal{L} be a collection of subgroups L of Γ such that $V^L \neq 0$. Here $V^L = \{v \in V : \gamma \cdot v = v \text{ for all } \gamma \in L\}$ is the fixed point set

of L in V . For every $L \in \mathcal{L}$, let $\text{Ind}_L^\Gamma : R(L) \rightarrow R(\Gamma)$ be the induction map and define

$$(5.9) \quad R_{\text{ind}}(\Gamma) = \sum_{L \in \mathcal{L}} \text{Ind}_L^\Gamma(R(L)) \subset R(\Gamma),$$

$$(5.10) \quad \overline{R}(\Gamma) = R(\Gamma)/R_{\text{ind}}(\Gamma).$$

The quotient $\overline{R}(\Gamma)$ is called the *elliptic representation ring* of Γ .

Define a bilinear form $\langle \cdot, \cdot \rangle_\Gamma$ on $R(\Gamma)$:

$$\langle \sigma_1, \sigma_2 \rangle_\Gamma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \text{tr}_{\sigma_1}(\gamma) \overline{\text{tr}_{\sigma_2}(\gamma)} = \dim_{\mathbb{C}} \text{Hom}_\Gamma(\sigma_1, \sigma_2),$$

where $\text{tr}_{\sigma_i}(\gamma)$ is the value of the character of σ_i on γ .

Define an elliptic pairing e_Γ on $R(\Gamma)$:

$$e_\Gamma(\sigma_1, \sigma_2) = \sum_{i=0}^{\dim_{\mathbb{C}} V} (-1)^i \dim_{\mathbb{C}} \text{Hom}_\Gamma(\sigma_1 \otimes \wedge^i V, \sigma_2) = \langle \sigma_1 \otimes \wedge^\pm V, \sigma_2 \rangle_\Gamma,$$

where $\wedge^\pm V = \sum_i (-1)^i \wedge^i V$.

It is shown in [11] that the radical $\text{rad}(e_\Gamma)$ of e_Γ is precisely $R_{\text{ind}}(\Gamma)$. Thus the elliptic pairing e_Γ induces a non-degenerate bilinear form on $\overline{R}(\Gamma)$, which is still denoted e_Γ , and furthermore

$$(5.11) \quad \dim_{\mathbb{C}} \overline{R}(\Gamma) = |\mathcal{C}_{\text{ell}}(\Gamma)|,$$

where $\mathcal{C}_{\text{ell}}(\Gamma)$ is the set of elliptic conjugacy classes in Γ .

5.2. Elliptic representations of real reflection groups. We now specialize to $\Gamma = W$ a finite real reflection group and V the reflection representation associated to W . We give two general properties about the elliptic representations of W .

A subgroup L of W is a proper parabolic subgroup if $L = \langle s_{\alpha_i} : \alpha_i \in J \rangle$ for some $J \subsetneq \Delta$. Let \mathcal{W} be the set of proper parabolic subgroups of W .

Proposition 5.1. *For any $L \in \mathcal{L}$ (notation in Section 5.1), there exists a proper parabolic subgroup P of W and $w \in W$ such that $L \subseteq wPw^{-1}$. In particular,*

$$(5.12) \quad R_{\text{ind}}(W) = \sum_{P \in \mathcal{W}} \text{Ind}_P^W(R(P)).$$

Proof. Let $L \in \mathcal{L}$ and let $v \in V^L$. There exists an element $w \in W$ such that $w \cdot v$ is in the fundamental chamber of V^\vee (i.e. $\text{Re}(\alpha_i, v) \geq 0$ for all simple roots α_i). Then the fixed point subgroup of $w \cdot v$ is the proper parabolic subgroup P generated by simple reflections fixing $w \cdot v$. Hence $L \subset w^{-1}Pw$ as desired. Since $\text{Ind}_L^W(R(L)) \subseteq \text{Ind}_{w^{-1}Pw}^W(R(w^{-1}Pw)) = \text{Ind}_P^W(R(P))$, we have the second assertion.

Q.E.D.

Recall that W' is defined in Section 2.2.

Proposition 5.2. *Let w be an elliptic element in W . Then $\det_V(w) = (-1)^r$, where $r = \dim_{\mathbb{C}} V$. In particular, the set of elliptic elements is a subset of W' .*

Proof. Realize w as a matrix in $O(V)$. Since w is a real matrix, ζ is an eigenvalue of w if and only if the complex conjugate of ζ is also an eigenvalue of w . Hence, the number of real eigenvalues of w has the same parity as r . Since w is elliptic, the only possible real eigenvalue of w is -1 . Hence $\det(w) = (-1)^r$.

Q.E.D.

Remark 5.3. We record the number of elliptic conjugacy classes in each noncrystallographic case:

$$I_2(n), n \text{ odd} : (n-1)/2, \quad I_2(n), n \text{ even} : n/2, \quad H_3 : 4, \quad H_4 : 20.$$

This also gives the dimension of $\overline{R}(W)$ by (5.11).

5.3. An isometry between $\overline{R}(W)$ and $R_{\text{gen}}(\widetilde{W})$. Let $R_{\text{gen}}(\widetilde{W})$ be the subspace of $R(\widetilde{W})$ spanned by genuine irreducible representations of \widetilde{W} .

Let \mathcal{C}^0 be the set of conjugacy classes C in W such that $p^{-1}(C)$ splits into two distinct conjugacy classes in \widetilde{W} . Then

$$(5.13) \quad \dim_{\mathbb{C}} R_{\text{gen}}(\widetilde{W}) = |\mathcal{C}(\widetilde{W})| - |\mathcal{C}(W)| = |\mathcal{C}^0|,$$

where $\mathcal{C}(W)$ (resp. $\mathcal{C}(\widetilde{W})$) is the set of conjugacy classes in W (resp. in \widetilde{W}).

Remark 5.4. The cardinality $|\mathcal{C}^0|$ in each case is as follows:

$$I_2(n), n \text{ odd} : (n+3)/2, \quad I_2(n), n \text{ even} : n/2, \quad H_3 : 8, \quad H_4 : 20.$$

Recall that the spin representation S of W is defined in Section 2.4. Define a map

$$i_S : R(W) \rightarrow R_{\text{gen}}(\widetilde{W}), \quad \sigma \mapsto \sigma \otimes S.$$

Proposition 5.5. *Let R be a noncrystallographic root system. The map i_S is surjective if and only if R is $I_2(n)$ with n even, H_3 or H_4 .*

Proof. Since the genuine \widetilde{W} representations are determined by the values of their characters on the conjugacy classes in $p^{-1}(\mathcal{C}_0)$, it suffices to check that the character of S does not vanish on all conjugacy classes in $p^{-1}(\mathcal{C}^0)$ if and only if R is not $I_2(n)$ with n odd. This can be verified by using Remark 2.2 with Table 1, Table 2 for $I_2(n)$ and Table 4 for H_3 and [9, Table II(b)] for H_4 .

Q.E.D.

The above proposition for the crystallographic root systems is shown in [4, Lemma 3.3.3].

We define some new notation. Let $R_{\text{gen}}(\widetilde{W}')$ be the subspace of $R(\widetilde{W}')$ generated by genuine representations of \widetilde{W}' . Define another map

$$i_S^{\pm} : R(W) \rightarrow R_{\text{gen}}(\widetilde{W}'), \quad \sigma \mapsto \sigma \otimes (S^+ - S^-).$$

It has been discovered in [5] that the map i_S^{\pm} is more interesting to be studied and is related to the Dirac index of modules of graded affine Hecke algebras.

Proposition 5.6. *The map i_S^\pm satisfies*

$$2e_W(\sigma_1, \sigma_2) = \langle i_S^\pm(\sigma_1), i_S^\pm(\sigma_2) \rangle_{\widetilde{W}'},$$

for $\sigma_1, \sigma_2 \in R(W)$.

Proof. This is [5, Proposition 3.1]. Indeed, it follows from the computation below:

$$\begin{aligned} \langle i^\pm(\sigma_1), i^\pm(\sigma_2) \rangle_{\widetilde{W}'} &= \langle \sigma_1 \otimes (S^+ - S^-) \otimes (S^+ - S^-)^*, \sigma_2 \rangle_{\widetilde{W}'}, \\ &= \frac{2}{[W : W']} \langle \sigma_1 \otimes \wedge^\pm V, \sigma_2 \rangle_{\widetilde{W}'}, \\ &= \frac{2}{[W : W']} \langle \sigma_1 \otimes \wedge^\pm V, \sigma_2 \rangle_W, \\ &= \frac{2}{|W|} \sum_{w \in W'} \text{tr}_{\sigma_1}(w) \text{tr}_{\wedge^\pm}(w) \overline{\text{tr}_{\sigma_2}(w)} \\ &= \frac{2}{|W|} \sum_{w \in W} \text{tr}_{\sigma_1}(w) \text{tr}_{\wedge^\pm}(w) \overline{\text{tr}_{\sigma_2}(w)}. \end{aligned}$$

In the first equality, $(S^+ - S^-)^*$ is the dual of $S^+ - S^-$. The second equality follows from Lemma 2.1. The remaining nontrivial equality is the fifth one. By [11, Lemma 2.1.1], $\text{tr}_{\wedge^\pm V}(w) = \det_V(1 - w)$ for any $w \in W$ and so $\text{tr}_{\wedge^\pm V}$ vanishes off the set of elliptic conjugacy classes in W . Now the fifth equality follows from Proposition 5.2. Thus we have

$$\langle i^\pm(\sigma_1), i^\pm(\sigma_2) \rangle_{\widetilde{W}'} = 2 \langle \sigma_1 \otimes \wedge^\pm V, \sigma_2 \rangle_W = 2e_W(\sigma_1, \sigma_2).$$

Q.E.D.

In general, the map i_S^\pm is not an injection since the elliptic pairing e_W may be degenerate. To remedy this, we consider the elliptic representation ring $\overline{R}(W) = R(W)/\text{rad}(e_W)$. Then the map i_S^\pm descends to an injection from $\overline{R}(W)$ to $R_{\text{gen}}(\widetilde{W}')$ by Proposition 5.6. Then we also have

$$(5.14) \quad \dim_{\mathbb{C}} \text{im}(i_S^\pm) = |\mathcal{C}_{\text{ell}}(W)|.$$

We now define an involution ι on $R_{\text{gen}}(\widetilde{W}')$. When $\dim_{\mathbb{C}} V$ is odd, define $\iota(\tilde{\sigma}) = \text{sgn} \otimes \tilde{\sigma}$. When $\dim_{\mathbb{C}} V$ is even, there is an outer automorphism of order 2 on \widetilde{W}' by the conjugation of an element in $\widetilde{W} - \widetilde{W}_{\text{even}}$. Then ι is defined to be the involution on $R_{\text{gen}}(\widetilde{W}')$ induced from the outer automorphism. In particular, we have $\iota(S^\pm) = S^\mp$, and

$$\iota(\sigma \otimes (S^+ - S^-)) = -\sigma \otimes (S^+ - S^-) \text{ for } \sigma \in R(W).$$

Thus $\text{im}(i_S^\pm) \subseteq \ker(\iota + 1)$. To study when another inclusion $\text{im}(i_S^\pm) \supseteq \ker(\iota + 1)$ holds, we need a simple lemma.

Let $\tilde{\sigma} \in \widetilde{W}$. First consider $\dim_{\mathbb{C}} V$ is even. We divide into two cases. If the restriction of $\tilde{\sigma}$ to \widetilde{W}' splits into two representations, then let $\tilde{\sigma}^+$ and $\tilde{\sigma}^-$ be those two representations (the choice is arbitrary). Otherwise the restriction of $\tilde{\sigma}$ to \widetilde{W}' is irreducible and let $\tilde{\sigma}^+ = \tilde{\sigma}^- = \tilde{\sigma}$. For $\dim_{\mathbb{C}} V$ odd, let $\tilde{\sigma}^+ = \tilde{\sigma}$ and $\tilde{\sigma}^- = \text{sgn} \otimes \tilde{\sigma}$. By definition, we have $\iota(\tilde{\sigma}^\pm) = \sigma^\mp$ no matter $\dim_{\mathbb{C}} V$ is odd or even.

Lemma 5.7. *Let $\tilde{\sigma} \in \text{Irr}_{\text{gen}}(\widetilde{W})$.*

(1) *Assume $\dim_{\mathbb{C}} V$ is even. Then $\tilde{\sigma} = \text{sgn} \otimes \tilde{\sigma}$ if and only if $\tilde{\sigma}^+ \neq \tilde{\sigma}^-$. Moreover,*

$$\dim_{\mathbb{C}} \ker(\iota + 1) = |\{\tilde{\sigma} \in \text{Irr}_{\text{gen}}(\widetilde{W}) : \tilde{\sigma} = \text{sgn} \otimes \tilde{\sigma}\}|.$$

(2) *Assume $\dim_{\mathbb{C}} V$ is odd. Then $\tilde{\sigma} = \text{sgn} \otimes \tilde{\sigma}$ if and only if $\tilde{\sigma}^+ = \tilde{\sigma}^-$. Moreover,*

$$2 \dim_{\mathbb{C}} \ker(\iota + 1) = |\{\tilde{\sigma} \in \text{Irr}_{\text{gen}}(\widetilde{W}) : \tilde{\sigma} \neq \text{sgn} \otimes \tilde{\sigma}\}|.$$

Proof. We only consider $\dim_{\mathbb{C}} V$ is even. The case for $\dim_{\mathbb{C}} V$ odd is similar (and easier). Note that

$$(5.15) \quad \langle \tilde{\sigma}, \tilde{\sigma} \rangle_{\widetilde{W}} = \frac{1}{2} \langle \tilde{\sigma}|_{\widetilde{W}'}, \tilde{\sigma}|_{\widetilde{W}'} \rangle_{\widetilde{W}'} + \frac{1}{|\widetilde{W}|} \sum_{w \in \widetilde{W}_{\text{odd}}} |\text{tr}_{\tilde{\sigma}}(w)|^2,$$

where $\widetilde{W}_{\text{odd}} = p^{-1}(\det_{V_0}^{-1}(-1) \cap W)$. Then $\langle \tilde{\sigma}|_{\widetilde{W}'}, \tilde{\sigma}|_{\widetilde{W}'} \rangle_{\widetilde{W}'} = 1$ or 2 . Note $\langle \tilde{\sigma}|_{\widetilde{W}'}, \tilde{\sigma}|_{\widetilde{W}'} \rangle_{\widetilde{W}'} = 2$ if and only if $|\text{tr}_{\tilde{\sigma}}(w)| = 0$ for all $w \in \widetilde{W}_{\text{odd}}$. This proves the first assertion of (1). Let

$$B = \{\tilde{\sigma}^+ - \tilde{\sigma}^- : \tilde{\sigma} \in \text{Irr}_{\text{gen}}(\widetilde{W}) \text{ and } \tilde{\sigma}^+ \neq \tilde{\sigma}^-\}.$$

The second assertion will follow from the first assertion if we show B forms a basis for $\ker(\iota + 1)$. To this end, note that B spans $\ker(\iota + 1)$ since $\iota(\tilde{\sigma}^+) = \tilde{\sigma}^-$, and B is linearly independent by considering the inner product $\langle \cdot, \cdot \rangle_{\widetilde{W}'}$.

Q.E.D.

Proposition 5.8. *Let R be a noncrystallographic root system. Then $\text{im}(i_S^{\pm}) = \ker(1 + \iota)$.*

Proof. Since $\text{im}(i_S^{\pm}) \subset \ker(1 + \iota)$, it suffices to show that $\dim_{\mathbb{C}} \text{im}(i_S^{\pm}) = \dim_{\mathbb{C}} \ker(1 + \iota)$. We record the number $|\{\tilde{\sigma} \in \text{Irr}_{\text{gen}}(\widetilde{W}) : \tilde{\sigma} = \text{sgn} \otimes \tilde{\sigma}\}|$ for $\dim_{\mathbb{C}} V$ even:

$$I_2(n) \text{ } n \text{ odd} : (n-1)/2, \quad I_2(n) \text{ } n \text{ even} : n/2, \quad H_4 : 20.$$

The number $|\{\tilde{\sigma} \in \text{Irr}_{\text{gen}}(\widetilde{W}) : \tilde{\sigma} \neq \text{sgn} \otimes \tilde{\sigma}\}|$ for H_3 is 8. Then the result follows from (5.14), Remark 5.3 and Lemma 5.7.

Q.E.D.

We remark that the above proposition is not true for some crystallographic cases, for example A_n ($n \geq 5$).

5.4. An orthonormal basis, and a connection to the graded affine Hecke algebra.

In this subsection, we will look at the example $W = W(I_2(n))$ (n odd). We shall construct an orthonormal basis for $\overline{R}(W)$ such that every element in the basis is the sum of irreducible characters in \mathbb{Z} -coefficients. The significance of this basis is explained in Remark 5.9 below.

Example on $I_2(n)$, n odd

Let $W = W(I_2(n))$. It is easy to see that \widetilde{W}' is a cyclic group of order $2n$. Then each genuine \widetilde{W} -representation $\tilde{\rho}_i$ in Table 1 splits into two one-dimensional \widetilde{W}' -representations, denoted $\tilde{\rho}_i^+$ and $\tilde{\rho}_i^-$. We shall fix $\tilde{\rho}_i^+$ and $\tilde{\rho}_i^-$ such that

$$\tilde{\rho}_i^+(f_{\alpha_1} f_{\alpha_2}) = -e^{2i\sqrt{-1}\pi/n} \text{ and } \tilde{\rho}_i^-(f_{\alpha_1} f_{\alpha_2}) = -e^{-2i\sqrt{-1}\pi/n}.$$

The spin module S of \widetilde{W} is the two dimensional module $\widetilde{\rho}_{(n-1)/2}$. Set $S^+ = \widetilde{\rho}_{(n-1)/2}^+$ and $S^- = \widetilde{\rho}_{(n-1)/2}^-$. Now a computation gives

$$\text{sgn} \otimes (S^+ - S^-) = S^+ - S^-,$$

and for $j = 1, \dots, (n-3)/2$,

$$\phi_j \otimes (S^+ - S^-) = \left(\widetilde{\rho}_{(n-1-2j)/2}^+ - \widetilde{\rho}_{(n-1-2j)/2}^- \right) - \left(\widetilde{\rho}_{(n+1-2j)/2}^+ - \widetilde{\rho}_{(n+1-2j)/2}^- \right).$$

where sgn and ϕ_j are as in Table 1 and considered to be the restriction in \widetilde{W}' (note that sgn restricted to \widetilde{W}' is just the trivial representation). For notational convenience, set $\widetilde{\sigma}_j = \widetilde{\rho}_{(n+1-2j)/2}^+ - \widetilde{\rho}_{(n+1-2j)/2}^-$ for $j = 1, \dots, (n-1)/2$.

By Proposition 5.1,

$$R_{\text{ind}}(W) = \text{span}_{\mathbb{C}} \{ \text{triv} \oplus \phi_1 \oplus \dots \oplus \phi_{(n-1)/2}, \text{sgn} \oplus \phi_1 \oplus \dots \oplus \phi_{(n-1)/2} \}$$

and hence $\{ \text{sgn}, \phi_1, \dots, \phi_{(n-3)/2} \}$ forms an ordered basis for $\overline{R}(W)$. We also fix an ordered basis for $\text{im}(i_S^\pm) = \ker(1 + \iota)$ (Proposition 5.8):

$$\{ \widetilde{\sigma}_1, \widetilde{\sigma}_2, \dots, \widetilde{\sigma}_{(n-1)/2} \}.$$

Then the matrix representation of i_S^\pm with respect to the above two bases is an upper triangular matrix $I - J$, where I is the $(n-1)/2 \times (n-1)/2$ identity matrix and J is the $(n-1)/2 \times (n-1)/2$ matrix with 1 in the superdiagonal and 0 elsewhere. Then inspecting the columns of the matrix $(I - J)^{-1} = I + J + \dots + J^{(n-3)/2}$, we could obtain a new basis $\{ \omega_i \}_{i=1}^{(n-1)/2}$ for $\overline{R}(W)$:

$$\omega_1 = \text{sgn}, \quad \omega_i = \text{sgn} \oplus \phi_1 \oplus \dots \oplus \phi_{i-1} \quad (i = 2, \dots, (n-1)/2),$$

so that the new basis has the property that $i_S^\pm(\omega_i) = \widetilde{\sigma}_i$. Then, by Proposition 5.6,

$$e_W(\omega_i, \omega_j) = \frac{1}{2} \langle i_S^\pm(\omega_i), i_S^\pm(\omega_j) \rangle_{\widetilde{W}'} = \langle \widetilde{\sigma}_i, \widetilde{\sigma}_j \rangle_{\widetilde{W}'} = \delta_{ij}.$$

Hence $\{ \omega_i \}_{i=1}^{(n-1)/2}$ is an orthonormal basis for $\overline{R}(W)$.

Remark 5.9. The importance of this orthonormal basis is as follows. Let \mathbb{H} be the graded affine Hecke algebra (see [1] for the definition) associated to W with the parameter function c . Let X and X' be discrete series of \mathbb{H} (i.e. for the central character γ of X (or X'), $(\omega, \gamma) < 0$ for all the fundamental weight ω in R). If W is a Weyl group, then the discrete series as W -modules form an orthonormal set in $\overline{R}(W)$ ([16, Section 3], also see [5]). We conjecture the same holds in noncrystallographic cases and so the above constructed orthogonal basis would give some hints for the unknown W -module structure of discrete series in noncrystallographic cases. Some study for discrete series in noncrystallographic cases can be found in [12], [14] and [15].

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